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SOME ASPECTS OF THE CATEGORY OF SUBOBJECTS OF CONSTANT OBJECTS IN A TOPOS

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Let $\mathscr{F}$ be a topos and let $\mathscr{E}$ be a topos defined over $\mathscr{F}$ by a geometric morphism γ. Objects of $\mathscr{E}$ of the form γ^*X for $X \in \mathscr{F}$ are called *constant* objects. In this paper we shall study the full subcategory $\mathscr{E}^*$ of $\mathscr{E}$ consisting of all *subobjects* of constant objects in $\mathscr{E}$. In the case where $\mathscr{F}$ is the category $\mathscr{S}$ of sets we construct, for each complete Heyting algebra H, a simple category $\tilde{H}$ which we show to be equivalent to $\mathscr{E}^*$ when H is the algebra of subobjects of the terminal object in $\mathscr{E}$. This yields a new and especially straightforward proof of the well-known result that a topos defined over $\mathscr{S}$ is equivalent as a category to a Boolean extension of the universe of sets iff it satisfies the axiom of choice. We go on to investigate the properties of $\tilde{H}$ and in Section 2 we extend some of our results to the case in which $\mathscr{F}$ is an arbitrary base topos.

1. Toposes defined over the category of sets

Let $\mathscr{E}$ be a topos defined over the category $\mathscr{S}$ of sets by a geometric morphism γ. In this case we know that $\gamma^*I = \coprod_I 1$ and $\gamma_* X = \mathscr{E}(1, X)$ for $I \in \mathscr{S}$, $X \in \mathscr{E}$. Moreover, the coproduct of any family of subobjects of 1 always exists in E (cf. the remark on page 120 of [3]), and the objects of $\mathscr{E}^*$ are precisely the objects of $\mathscr{E}$ which are of this form. We first find a particularly simple alternative description of $\mathscr{E}^*$.

Let H be a complete Heyting algebra (frame, locale). We define the category $\tilde{H}$ as follows. The *objects* of $\tilde{H}$ are all functions $I \xrightarrow{a} H$, $a = \langle a_i \rangle_{i \in I}$ for all sets I. If $I \xrightarrow{a} H$, $J \xrightarrow{b} H$ are two objects in $\tilde{H}$, an *arrow* $a \xrightarrow{p} b$ is a function $p: I \times J \to H$, $p = \langle p_{ij} \rangle_{i \in I, j \in J}$ such that

$$p_{ij} \le b_j \qquad (i \in I, j \in J), \tag{1.1}$$

$$p_{ij} \wedge p_{ij'} = 0 \quad (i \in I, j \ne j' \in J), \tag{1.2}$$

$$\bigvee_{j \in J} p_{ij} = a_i \qquad (i \in I). \tag{1.3}$$

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(We may think of an object a of $\tilde{H}$ as an 'H-valued set' in which $a(i) \in H$ is the 'H-value' of the statement $i \in a$. An arrow $a \xrightarrow{p} b$ in $\tilde{H}$ may be thought of as an 'H-valued functional relation' between a and b.) If $c = \langle c_k \rangle_{k \in K}$ is an object of $\tilde{H}$ and $q: J \times K \to H$ is an arrow $b \to c$ in $\tilde{H}$, the composition $qp = r$ of p and q is defined by

$$r_{ik} = \bigvee_{j \in J} p_{ij} \wedge q_{jk}.$$

It is easy to check that composition is associative and that the identity arrow $\mathrm{id}: a \to a$ is given by the 'Kronecker delta' function $\delta: I \times I \to H$ such that

$$\delta_{ii'} = 0 \quad (i \neq i'), \qquad \delta_{ii} = 1.$$

If $\mathscr{E}$ is an $\mathscr{S}$-topos, then (cf. the proof of 5.37 of [3]), $\gamma_* \Omega_{\mathscr{E}}$ is a complete Heyting algebra; it is naturally isomorphic to the (partially ordered) set of subobjects of 1 in $\mathscr{E}$. Thus the latter is a complete Heyting algebra.

Now we can prove

1.1. Theorem. *Let $\mathscr{E}$ be an $\mathscr{S}$-topos, and let H be the complete Heyting algebra of subobjects of* 1 *in $\mathscr{E}$. Then $\mathscr{E}^* \simeq \tilde{H}$. If in addition the axiom of choice holds in $\mathscr{E}$, then* (H *is a complete Boolean algebra and*) $\mathscr{E} \simeq \tilde{H}$.

Proof. We define a functor $F: \tilde{H} \to \mathscr{E}$ as folows. For each object $a: I \to H$ in $\tilde{H}$ we put

$$F(a) = \coprod_{i \in I} a_i.$$

If $b: J \to H$ is an object in $\tilde{H}$ and $p: a \to b$ an arrow in $\tilde{H}$, we define $F(p): F(a) \to F(b)$ as follows. From (1.2) and (1.3) we have

$$a_i \cong \coprod_{j \in J} p_{ij} \quad (i \in I)$$

and from the (unique) arrows $p_{ij} \rightarrowtail b_j$ given by (1.1) we obtain for each $i \in I$ a unique arrow s_i such that the diagram

$$\begin{array}{ccc} p_{ij} & \rightarrowtail & b_j \\ \downarrow & & \downarrow \\ \coprod_{j \in J} p_{ij} & \xrightarrow{s_i} & \coprod_{j \in J} b_j \end{array}$$

commutes for all $i \in I$, $j \in J$, where the downward arrows are canonical injections. We put p_i' for the composition

$$a_i \xrightarrow{\sim} \coprod_{j \in J} p_{ij} \xrightarrow{s_i} \coprod_{j \in J} b_j.$$

Thus p_i' is the unique arrow making the diagram

$$\begin{array}{ccc} p_{ij} & \rightarrowtail & b_i \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{p_i'} & \coprod b_j \end{array} \tag{1.4}$$

commute. We finally define $F(p)$ to be the unique arrow such that the diagram

$$\begin{array}{ccc} a_i & \xrightarrow{\sigma_i} & \coprod\limits_{i\in I} a_i \\ & {\scriptstyle p_i'}\searrow & \downarrow{\scriptstyle F(p)} \\ & & \coprod\limits_{j\in J} b_j \end{array}$$

commutes for each $i \in I$, where σ_i is the canonical injection.

It is not hard to check that F is a functor, and clearly each object in $\mathscr{E}^*$ is (isomorphic to an object) in the range of F. Accordingly, to show that F is an equivalence it suffices to show that F is full and faithful.

To verify the fidelity of F, we first observe that the diagram (1.4) is a pullback for each $i \in I$, $j \in J$. For let

$$\begin{array}{ccc} r_{ij} & \rightarrowtail & b_j \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{p_i'} & \coprod b_j \end{array}$$

be a pullback. Then clearly, since (1.4) commutes, we have $p_{ij} \leq r_{ij}$. On the other hand, by the universality of coproducts in $\mathscr{E}$, we have

$$a_i \cong \coprod_{j\in J} r_{ij},$$

so that

$$\bigvee_{j\in J} p_{ij} = a_i = \bigvee_{j\in J} r_{ij}.$$

But it now follows from the disjointness of coproducts in $\mathscr{E}$ that $r_{ij} \wedge r_{ij'} = 0$ when $j \neq j'$. One easily concludes from this that $p_{ij} = r_{ij}$, so (1.4) is indeed a pullback.

Now let $a \xrightarrow{p} b$ and $a \xrightarrow{q} b$ be arrows in $\tilde{H}$ and suppose that $F(p) = F(q)$. Then $p_i' = q_i'$ for all $i \in I$ and so, since (1.4) is a pullback, it follows that $q_{ij} \leq p_{ij}$. Similarly, $p_{ij} \leq q_{ij}$ and so $p = q$. Hence F is faithful as claimed.

Finally, we show that F is full. Suppose that a, b are objects in $\tilde{H}$ and that

$F(a) \xrightarrow{f} F(b)$ is an arrow in $\mathscr{E}$. For each i, j form the pullback

$$\begin{array}{ccc} p_{ij} & \rightarrowtail & b_j \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{f\sigma_i} & \coprod b_j \end{array} \tag{1.5}$$

By the universality of coproducts in $\mathscr{E}$ we have $\coprod_{j\in J} p_{ij} \cong a_i$ and by the disjointness of coproducts in $\mathscr{E}$ we have $p_{ij} \wedge p_{ij'} = 0$ for $j \neq j'$, whence $\bigvee_{j\in J} p_{ij} = a_i$. Hence $p = \langle p_{ij} \rangle_{i\in I, j\in J}$ is an arrow $a \to b$ in $\tilde{H}$. We claim $F(p) = f$. For this to be the case it suffices that $F(p)\sigma_i = f\sigma_i$ for all $i \in I$. But this follows immediately from (1.4), (1.5) and the fact that $p_i' = F(p)\sigma_i$.

Thus F is an equivalence and $\mathscr{E}^* \simeq \tilde{H}$.

Now suppose that $\mathscr{E}$ satisfies the axiom of choice. Then, by 5.3 of [3], the subobjects of 1 form a set of generators in $\mathscr{E}$ and so each object of $\mathscr{E}$ is covered by a family of subobjects of 1. Using the axiom of choice in $\mathscr{E}$, it follows easily from this that each object of $\mathscr{E}$ is isomorphic to a coproduct of subobjects of 1, whence $\mathscr{E} = \mathscr{E}^* \simeq \tilde{H}$. □

We recall that [1] that, for each complete Boolean algebra B, the Boolean extension $V^{(B)}$ of the universe of sets in the sense of Scott–Solovay may be regarded as an $\mathscr{S}$-topos in a natural way. Since the axiom of choice holds in such a topos (provided it holds in $\mathscr{S}$!), Theorem 1.1 yields as an immediate consequence the following well-known result.

1.2. Corollary. *An $\mathscr{S}$-topos is equivalent to one of the form $V^{(B)}$ for a complete Boolean algebra B if and only if it satisfies the axiom of choice.*

Our next theorem shows that a number of conditions on $\mathscr{E}^*$ and H are equivalent.

1.3. Theorem. *Let $\mathscr{E}$ be an $\mathscr{S}$-topos and let H be the complete Heyting algebra of subjects of* 1 *in $\mathscr{E}$. Consider the conditions:*

(i) *$\mathscr{E}$ satisfies the axiom of choice;*

(ii) *$\mathscr{E}^* \hookrightarrow \mathscr{E}$ is an equivalence;*

(iii) *$\Omega_{\mathscr{E}}$ is isomorphic to an object in $\mathscr{E}^*$;*

(iv) *$\mathscr{E}$ is Boolean;*

(v) *$\mathscr{E}^* \simeq \tilde{H}$ is a topos;*

(vi) *$\mathscr{E}^* \simeq \tilde{H}$ has a subobject classifier;*

(vii) *H is a Boolean algebra;*

(viii) *$\tilde{H} \simeq V^{(B)}$ for some complete Boolean algebra B.*

Then (i) ⇔ (ii) ⇒ (iii) ⇔ (iv) ⇒ (v), *and* (v) *through* (viii) *are equivalent. If $\mathscr{E}$ is* localic *over $\mathscr{S}$, then all the conditions are equivalent. Thus conditions* (v) *through* (viii) *are equivalent for any complete Heyting algebra H.*

Proof. (i)$\Rightarrow$(ii) follows from Theorem 1.1.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv). Recall that an object X of a topos is said to be *decidable* if the diagonal subobject $X \overset{\Delta}{\rightarrowtail} X \times X$ has a complement. It is easy to verify that, since each object X of $\mathscr{S}$ is decidable, so is each object of $\mathscr{E}$ of the form γ^*X and hence so is any subobject of such an object; i.e. any object in $\mathscr{E}^*$ is decidable. Thus condition (iii) implies that $\Omega_{\mathscr{E}}$ is decidable, and this is well known to be equivalent to Booleanness of $\mathscr{E}$.

(iv)$\Rightarrow$(iii). If $\mathscr{E}$ is Boolean, then $\Omega_{\mathscr{E}} \cong 1+1 \cong \gamma^*(1+1) \in \mathscr{E}^*$.

(ii)$\Rightarrow$(i). If (ii) holds, then $\mathscr{E}$ is certainly localic over $\mathscr{S}$; but since (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) $\mathscr{E}$ is also Boolean. Then by 5.39 of [3] the axiom of choice in $\mathscr{S}$ yields the axiom of choice in $\mathscr{E}$.

(iii)$\Rightarrow$(vi). Since $\mathscr{E}^*$ is easily seen to be closed under products and subobjects in $\mathscr{E}$, it follows that it is also closed under pullbacks in $\mathscr{E}$. The implication in question now follows easily.

(v)$\Rightarrow$(vi) is trivial.

(vi)$\Rightarrow$(vii). Let $\langle b_j \rangle_{j\in J}$ be the subobject classifier in $\tilde{H}$. Then, given $a \in H$, the object $\langle a \rangle$ of $\tilde{H}$ is a subobject of the terminal object $\langle 1 \rangle$ in $\tilde{H}$ and so there are arrows

$$\langle 1 \rangle \xrightarrow{p} \langle b_j \rangle_{j\in J}, \qquad \langle 1 \rangle \xrightarrow{q} \langle b_j \rangle_{j\in J}$$

in $\tilde{H}$ such that

$$\begin{array}{ccc} \langle a \rangle & \rightarrowtail & \langle 1 \rangle \\ \downarrow & & \downarrow{\scriptstyle q} \\ \langle 1 \rangle & \overset{p}{\rightarrowtail} & \langle b_j \rangle_{j\in J} \end{array} \tag{1.6}$$

is a pullback. Since p and q are arrows in $\tilde{H}$, we have

$$p_j \wedge p_k = q_j \wedge q_k = 0 \quad \text{for } j \neq k \in J,$$

$$1 = \bigvee_{j\in J} p_j = \bigvee_{k\in J} q_k,$$

so that

$$\bigvee_{j\in J} \bigvee_{k\in K} p_j \wedge q_k = 1. \tag{1.7}$$

Since (1.6) commutes, we have

$$a \wedge q_j = a \wedge p_j \quad (j \in J),$$

so that

$$\begin{aligned} a = a\wedge 1 &= \bigvee_{j\in J} a\wedge p_j \\ &= \bigvee_{j\in J} a\wedge p_j \wedge q_j \\ &\le \bigvee_{j\in J} p_j\wedge q_j. \end{aligned} \tag{1.8}$$

But since (1.6) is a pullback, we must have, for all $c\in H$,

$$\forall j\in J\,[c\wedge p_j = c\wedge q_j] \;\Rightarrow\; c\le a.$$

In particular, taking $c=\bigvee_{j\in J} p_j\wedge q_j$, we get

$$\bigvee_{j\in J} p_j\wedge q_j \le a,$$

so that, by (1.8), $a=\bigvee_{j\in J} p_j\wedge q_j$. But then, by (1.7), a has a complement $\bigvee_{j\neq k} p_j\wedge q_k$ in H. This gives (vii).

(vii)$\Rightarrow$(viii). This follows from Theorem 1.1.

(viii)$\Rightarrow$(v) is trivial.

Finally, if $\mathscr{E}$ is localic over $\mathscr{S}$, then (vii)$\Rightarrow$(iv) and hence (in this case) (vii)$\Rightarrow$(i). For suppose that $\mathscr{E}$ is not Boolean; then $\neg\neg : \Omega\to\Omega$ is not the identity. Hence by the localicity of $\mathscr{E}$ there is $U\rightarrowtail 1$ and $U\xrightarrow{\alpha}\Omega$ such that

$$U\xrightarrow{\alpha}\Omega \neq U\xrightarrow{\alpha}\Omega\xrightarrow{\neg\neg}\Omega.$$

Since Ω is injective there is $1\xrightarrow{\beta}\Omega$ such that

$$U\xrightarrow{\alpha}\Omega = U\longrightarrow 1\xrightarrow{\beta}\Omega.$$

Clearly, then

$$1\xrightarrow{\beta}\Omega\xrightarrow{\neg\neg}\Omega \neq 1\xrightarrow{\beta}\Omega.$$

But this means that the subobject of 1 classified by β is not equal to its double complement in H, i.e. H is not Boolean. □

Remark. It is well known that the implication (i)$\Rightarrow$(iv) cannot be reversed; e.g. take $\mathscr{E}$ to be the topos $\mathscr{S}^G$ of G-sets for a group G. A similar counterexample shows the irreversibility of the implication (iv)$\Rightarrow$(v): take $\mathscr{E}$ to be the topos $\mathscr{S}^M$ of M-sets for a monoid M which is not a group. Then E is not Boolean; on the other hand 1 has only two subobjects 0 and 1 in $\mathscr{E}$, so $\mathscr{E}^*\simeq\mathscr{S}$ and (v) is satisfied.

2. Toposes defined over an arbitrary base topos

We now suppose that $\mathscr{E}$ is a topos defined over an arbitrary base topos $\mathscr{F}$ by a geometric morphism γ and investigate the extent to which the results and constructions of the previous section carry over to this more general setting. We shall employ freely the internal (Mitchell–Benabou) language of a topos as presented in §5.4 of [3].

To begin with, let us see how to generalize the construction of $\tilde{H}$. Let H be an internally complete Heyting algebra object in $\mathscr{F}$; we define the category $\tilde{H}$ as follows. (It is important to observe that $\tilde{H}$ is an 'honest-to-goodness' category, *not* an internal category in $\mathscr{F}$.)

First of all, the objects of $\tilde{H}$ are the objects of $\mathscr{F}/H$, i.e. all arrows $I \xrightarrow{a} H$ in $\mathscr{F}$.

Before defining the arrows of $\tilde{H}$ we need some notation. We let

$$\lambda_H : \Omega_{\mathscr{F}} \to H$$

be the arrow defined by

$$\lambda_H(p) = \bigvee_H \{a \in H : (a = 1_H) \wedge p\},$$

where p is a variable of type $\Omega_{\mathscr{F}}$. For each object J of $\mathscr{F}$, we let

$$\delta_J : J \times J \to \Omega_{\mathscr{F}}$$

be the classifying arrow of the diagonal subobject of $J \times J$, and we put eq_J for the composition

$$J \times J \xrightarrow{\delta_J} \Omega_{\mathscr{F}} \xrightarrow{\lambda_H} H.$$

Now we can define the arrows of $\tilde{H}$. Given objects $I \xrightarrow{a} H$ and $J \xrightarrow{b} H$ of $\tilde{H}$, an *arrow* $a \xrightarrow{p} b$ in $\tilde{H}$ is an arrow $I \times J \xrightarrow{p} H$ in $\mathscr{F}$ satisfying the following conditions, where i, j, j', x are variables of types I, J, J, H respectively:

$$F \models p(i, j) \leq_H b(j) \tag{2.1}$$

$$F \models p(i, j) \wedge p(i, j') \leq_H \mathrm{eq}_J(j, j') \tag{2.2}$$

$$F \models \bigvee_H \{x : \exists j\, [x = p(i, j)]\} = a(i). \tag{2.3}$$

(Notice that these conditions are just the 'internal' analogues of the conditions (1.1), (1.2), (1.3).) If $K \xrightarrow{c} H$ is an object of $\tilde{H}$ and $J \times K \xrightarrow{q} H$ is an arrow $b \longrightarrow c$ in $\tilde{H}$, the *composition* $qp = r$ is given by

$$r(i, k) = \bigvee_H \{x : \exists j\, [x = p(i, j) \wedge q(j, k)]\},$$

where k is a variable of type k. The identity arrow

$$a \xrightarrow{\mathrm{id}_a} a$$

in $\tilde{H}$ is given by

$$\mathrm{id}_a = \wedge_H \cdot \langle \mathrm{eq}_I, a\pi_1 \rangle,$$

where π_1 is 'projection onto the first coordinate' and $\wedge_H$ is the meet operation in H. It is readily checked that these data do determine a category.

Now let $\mathscr{E} \xrightarrow{\gamma} \mathscr{F}$ be a geometric morphism. Then ([3], 5.36) $H = \gamma_* \Omega_{\mathscr{E}}$ is an internally complete Heyting algebra object in $\mathscr{F}$, and in this case it is easily verified that the arrow $\lambda = \lambda_H$ has $(\gamma^* \dashv \gamma_*)$-transpose $\bar{\lambda} : \gamma^* \Omega_{\mathscr{F}} \to \Omega_{\mathscr{E}}$ classifying $\gamma^*(\mathrm{true}_{\mathscr{F}})$.

We recall that we have defined $\mathscr{E}^*$ to be the full subcategory of $\mathscr{E}$ whose objects are all subobjects of objects of the form $\gamma^* I$ for $I \in \mathscr{F}$. We shall prove the analogue of 1.1 in this more general context.

2.1. Theorem. $\mathscr{E}^* \simeq (\gamma_* \Omega_{\mathscr{E}})^{\sim}$.

Before giving the proof, we need some more terminology and a lemma.

Let $X \xrightarrow{f} Y$ be a partial arrow in $\mathscr{E}$, given by the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle f'} \big\downarrow & & \\ X & & \end{array} \tag{2.4}$$

We define the *graph* of f, $\mathrm{gph}(f)$, to be the image of the arrow

$$X' \xrightarrow{\langle f', f \rangle} X \times Y,$$

i.e. the extension of the formula

$$\exists x' \; [\langle x, y \rangle = \langle f'(x'), f(x') \rangle],$$

where x, x', y are variables of types X, X', Y respectively.

2.2. Lemma. *Let $X \times Y \xrightarrow{r} \Omega_{\mathscr{E}}$, let R be the subobject of $X \times Y$ classified by r, and let $\ulcorner R \urcorner$ be the corresponding global element of $\Omega_{\mathscr{E}}^{X \times Y}$. Then the following are equivalent:*

(i) $R = \mathrm{gph}(f)$ *for some* $X \xrightarrow{f} Y$;

(ii) $E \models \langle x, y \rangle \in \ulcorner R \urcorner \wedge \langle x, z \rangle \in \ulcorner R \urcorner \Rightarrow y = z$;

(iii) $E \models r(x, y) \wedge r(x, z) \leq_{\Omega_{\mathscr{E}}} \delta_Y(y, z)$,

where x, y, z are variables of type X, Y, Y respectively. Moreover, if these conditions hold, then the subobject X' of X on which f is defined may be taken to be $\| x \colon \exists y \, \langle x, y \rangle \in \ulcorner R \urcorner \|$, *or equivalently* $\| x \colon \exists y \, [r(x, y) = \mathrm{true}_{\mathscr{E}}] \|$.

Proof. (ii) $\Leftrightarrow$ (iii) holds by definition.

(i) ⇒ (ii). We have, introducing variables x', x'' of type X',

$$\begin{aligned}\mathscr{E} \vDash \langle x, y\rangle \in \ulcorner \mathrm{gph}(f)\urcorner \wedge \langle x, z\rangle \in \ulcorner \mathrm{gph}(f)\urcorner \\ \Rightarrow \exists x'\,[x = f'(x') \wedge y = f(x')] \wedge \exists x''\,[x = f'(x'') \wedge z = f(x'')] \\ \Rightarrow y = z,\end{aligned}$$

by the monicity of f' (see diagram (2.4)).

(ii) ⇒ (i). Form the pullback

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle \{\cdot\}} \\ X & \xrightarrow{|\{y:\langle x,y\rangle \in \ulcorner R \urcorner\}|} & \Omega^{Y} \end{array} \qquad (2.5)$$

Then we have

$$\begin{aligned}\mathscr{E} \vDash \langle x, y\rangle \in \ulcorner \mathrm{gph}(f)\urcorner &\Leftrightarrow \exists x'\,[x = f'(x') \wedge y = f(x')] \\ &\Leftrightarrow \{x\colon \langle x, z\rangle \in \ulcorner R\urcorner\} = \{y\} \quad \text{(since (2.5) is a pullback)} \\ &\Leftrightarrow \langle x, y\rangle \in \ulcorner R \urcorner \quad \text{(by (ii))}.\end{aligned}$$

Thus $R = \mathrm{gph}(f)$ as required.

To prove the final assertion, we merely observe that, by the above,

$$\mathscr{E} \vDash \exists x'\,[x = f'(x')] \Leftrightarrow \exists y\,[\langle x, y\rangle \in \ulcorner R\urcorner]. \qquad \square$$

Now we can provide the

Proof of Theorem 2.1. We define a functor

$$\beta : (\gamma_* \Omega_{\mathscr{E}})^{\sim} \to \mathscr{E}^*$$

as follows. Given an arrow $I \xrightarrow{a} \gamma_* \Omega_{\mathscr{E}}$ in $(\gamma_* \Omega_{\mathscr{E}})^{\sim}$, let

$$\gamma^* I \xrightarrow{\bar{a}} \Omega_{\mathscr{E}}$$

be its transpose across the adjunction $\gamma^* \dashv \gamma_*$, and let $\beta(a)$ be the subobject of $\gamma^* I$ classified by $\bar{a}$. Clearly $\beta(a) \in \mathscr{E}^*$ and every object of $\mathscr{E}^*$ is isomorphic to an object of this form.

Next, given an object $J \xrightarrow{b} \gamma_* \Omega_{\mathscr{E}}$ and an arrow $a \xrightarrow{p} b$ in $(\gamma_* \Omega_{\mathscr{E}})^{\sim}$, i.e. an arrow $I \times J \xrightarrow{p} \gamma_* \Omega_{\mathscr{E}}$ in $\mathscr{F}$ satisfying (2.1), (2.2), (2.3) (with $H = \gamma_* \Omega_{\mathscr{E}}$), let

$$\bar{p} : \gamma^* I \times \gamma^* J \cong \gamma^*(I \times J) \longrightarrow \Omega_{\mathscr{E}}$$

be its transpose across $\gamma^* \dashv \gamma_*$. After transposition across $\gamma^* \dashv \gamma_*$, conditions (2.1), (2.2) and (2.3) become the following, where x, y, z are variables of type $\gamma^* I$, $\gamma^* J$, $\gamma^* J$, respectively, and $\bar{a}, \bar{b}$ are the transposes of a, b, respectively:

$$\mathscr{E} \models \bar{p}(x, y) \leq \bar{b}(y) \tag{2.1'}$$

$$\mathscr{E} \models \bar{p}(x, y) \wedge \bar{p}(x, z) \leq \delta_{\gamma^* J}(y, z) \tag{2.2'}$$

$$|\exists y\ [\bar{p}(x, y) = \text{true}_{\mathscr{E}}]| = \bar{a}(x). \tag{2.3'}$$

From Lemma 2.2 we see that (2.2′) implies that there is a partial arrow

$$\gamma^* I \xrightarrow{f} \gamma^* J,$$

unique up to isomorphism, such that gph(f) is equal to the subobject of $\gamma^* I \times \gamma^* J$ classified by $\bar{p}$. Condition (2.3′) tells us that f is defined on the subobject $\beta(a)$ of $\gamma^* I$ classified by $\bar{a}$, and (2.1′) that the image of f is contained in the subobject $\beta(b)$ of $\gamma^* J$ classified by $\bar{b}$. Thus we may regard f as an arrow

$$\beta(a) \xrightarrow{f} \beta(b).$$

We put $\beta(p) = f$.

One can now check (tediously!) that β preserves composition and the identity arrows. Thus we have a functor

$$\beta : (\gamma^* \Omega_{\mathscr{E}})^{\sim} \longrightarrow \mathscr{E}^*.$$

It remains to show that β is an equivalence of categories. We have already remarked that every object in $\mathscr{E}^*$ is isomorphic to one in the range of β. Also, β is clearly faithful. To show that β is full, let $a, b \in (\gamma_* \Omega_{\mathscr{E}})^{\sim}$ and let $\beta(a) \xrightarrow{f} \beta(b)$ be an arrow in $\mathscr{E}^*$. Let $\bar{p}$ be the characteristic arrow of the subobject of $\gamma^* I \times \gamma^* J$ corresponding to the graph of f. It is then easy to check that (2.1′), (2.2′), (2.3′) hold for $\bar{p}$, and transposition across $\gamma^* \dashv \gamma_*$ yields (2.1), (2.2), (2.3) for its transpose p. Thus $a \xrightarrow{p} b$ is an arrow in $(\gamma_* \Omega_{\mathscr{E}})^{\sim}$, and clearly $\beta(p) = f$. Hence β is full, and therefore and equivalence. □

By taking $\mathscr{E} = \mathscr{F}$ and γ the identity functor in Theorem 2.1, we immediately obtain

2.3. Corollary. $\tilde{\Omega}_{\mathscr{F}} \simeq \mathscr{F}$. □

Having affirmed that Theorem 1.1 carries over to the case of an arbitrary base topos, we may now ask to what extent the same is true of Theorem 1.3. I have not been able to solve this problem completely, but I shall give a partial solution in Theorem 2.5. First, however, we require another lemma, which gives a canonical representation of objects in $\mathscr{E}^*$.

2.4. Lemma. *For each object X of $\mathscr{E}$ the following are equivalent:*

(i) $X \in \mathscr{E}^*$;

(ii) *there is a monic* $X \to \gamma^*\gamma_*\tilde{X}$, *where* $\tilde{X}$ *is the partial map classifier of* X. *If* X *is* injective, *then* $\tilde{X}$ *may be replaced by* X.

Proof. (ii) ⇒ (i) being trivial, we need only prove (i) ⇒ (ii). If $X \in \mathscr{E}^*$, then by definition there is $Y \in \mathscr{F}$ and a monic $X \rightarrowtail \gamma^*Y$. Hence there is $\gamma^*Y \xrightarrow{\alpha} \tilde{X}$ such that the diagram

$$\begin{array}{ccc} X & \overset{\eta}{\rightarrowtail} & \tilde{X} \\ \downarrow & \nearrow_{\alpha} & \\ \gamma^*Y & & \end{array}$$

commutes. Now if $Y \xrightarrow{\bar{\alpha}} \gamma_*\tilde{X}$ is the transpose of α across $\gamma^* \dashv \gamma_*$, we have the commutative diagram

$$\begin{array}{ccc} \gamma^*Y & \xrightarrow{\alpha} & \tilde{X} \\ {\scriptstyle \gamma^*(\bar{\alpha})}\downarrow & \nearrow_{\varepsilon} & \\ \gamma^*\gamma_*\tilde{X} & & \end{array}$$

where ε is the counit arrow. Hence the composition

$$X \rightarrowtail \gamma^*Y \xrightarrow{\gamma^*(\bar{\alpha})} \gamma^*\gamma_*\tilde{X}$$

is monic.

Clearly, if X is injective, we may replace $\tilde{X}$ by X and η by the identity arrow. □

Now we can prove:

2.5. Theorem. *Let* $\mathscr{F}$ *be a topos, let* $\mathscr{E}$ *be a topos defined over* $\mathscr{F}$ *by a geometric morphism* γ, *and let* $H = \gamma_*\Omega_{\mathscr{E}}$. *Consider the conditions:*

(i) $\Omega_{\mathscr{E}}$ *is isomorphic to an object in* $\mathscr{E}^*$;

(ii) *the counit arrow* $\gamma^*\gamma_*\Omega_{\mathscr{E}} \xrightarrow{\varepsilon} \Omega_{\mathscr{E}}$ *has a section;*

(iii) $\mathscr{E}$ *is Boolean;*

(iv) $\mathscr{E}^* \simeq \tilde{H}$ *has a subobject classifier;*

(v) H *is an internal Boolean algebra.*

Then (i) ⇔ (ii). *If* $\mathscr{F}$ *is Boolean, then* (i) ⇔ (iii) ⇒ (iv) ⇔ (v). *Finally, if* $\mathscr{F}$ *is Boolean and* $\mathscr{E}$ *is* localic *over* $\mathscr{F}$, *then all the conditions are equivalent.*

Proof. (ii) ⇒ (i) is trivial.

(i) ⇒ (ii). By Lemma 2.4, if (i) holds, then since $\Omega_{\mathscr{E}}$ is injective we have a monic $\Omega_{\mathscr{E}} \overset{\alpha}{\rightarrowtail} \gamma^*\gamma_*\Omega_{\mathscr{E}}$. Again using the injectivity of $\Omega_{\mathscr{E}}$, there is an arrow $\gamma^*\gamma_*\Omega_{\mathscr{E}} \xrightarrow{\beta} \Omega_{\mathscr{E}}$

such that $\beta \cdot \alpha = \mathrm{id}$. Let $\gamma_*\Omega_{\mathscr{E}} \xrightarrow{\bar{\beta}} \gamma_*\Omega_{\mathscr{E}}$ be the transpose of β across $\gamma^* \dashv \gamma_*$. Then $\beta = \varepsilon \cdot \gamma^*(\bar{\beta})$, so

$$\mathrm{id} = \beta \cdot \alpha = \varepsilon \cdot \gamma^*(\bar{\beta}) \cdot \alpha.$$

From now on we suppose that $\mathscr{F}$ is Boolean.

(i)$\Leftrightarrow$(iii) is proved in exactly the same manner as (iii)$\Leftrightarrow$(iv) in Theorem 1.3.

(i)$\Rightarrow$(iv) is proved in just the same way as (iii)$\Rightarrow$(vi) in Theorem 1.3.

(iv)$\Rightarrow$(v). Suppose that $\mathscr{E}^*$ has a subobject classifier

$$1 \xrightarrow{\text{true}'} \Omega'.$$

Since Ω' is a subobject of an object of the form γ^*X for some $X \in \mathscr{F}$ and since each object in the Boolean topos $\mathscr{F}$ is decidable, Ω' itself must be decidable. By standard arguments, the global element

$$1 \xrightarrow{\text{true}'} \Omega'$$

must then have a complement

$$1 \xrightarrow{\text{false}'} \Omega'.$$

It follows that

$$\begin{pmatrix} \text{true}' \\ \text{false}' \end{pmatrix}$$

is an isomorphism between $1+1$ and Ω'. Thus $1+1$ is the subobject classifier in $\mathscr{E}^*$.

Now, by §1 of [2], we may without loss of generality replace $\mathscr{E}$ by the *localic* topos $\mathscr{F}[H]$ of internal sheaves on H (since $\mathscr{E}^*$ is unaffected by the change). Since $\mathscr{F}[H]$ is localic over $\mathscr{F}$, we have a diagram of the form

$$\begin{array}{ccc} Y & \rightarrowtail & \gamma^*X \\ \downarrow & & \\ \Omega & & \end{array}$$

where $X \in \mathscr{F}$ and Ω is the subobject classifier in $\mathscr{F}[H]$. Since Ω is injective, we have an epic $\gamma^*X \twoheadrightarrow \Omega$. Form the pullback

$$\begin{array}{ccc} Z & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ \gamma^*X & \xrightarrow{\alpha} & \Omega. \end{array}$$

Since Z and γ^*X are both in $\mathscr{E}^*$ and $1+1$ is the subobject classifier in $\mathscr{E}^*$, we have a pullback (in $\mathscr{E}^*$, and hence also in $\mathscr{F}[H]$)

$$\begin{array}{ccc} Z & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \sigma_1} \\ \gamma^* X & \xrightarrow{\ \beta\ } & 1+1 \end{array}$$

where σ_1 is a canonical injection. Combining this with the obvious pullback diagram

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \sigma_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{true}} \\ 1+1 & \xrightarrow{\binom{\mathrm{true}}{\mathrm{false}}} & \Omega \end{array}$$

yields a pullback

$$\begin{array}{ccc} Z & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{true}} \\ \gamma^* X & \xrightarrow{\binom{\mathrm{true}}{\mathrm{false}}\cdot\beta} & \Omega \end{array}$$

But then α and $\beta\cdot\binom{\mathrm{true}}{\mathrm{false}}$ both classify the subobject Z of γ^*X, so they are equal; and since α is epic, so is $\binom{\mathrm{true}}{\mathrm{false}}$. But $\binom{\mathrm{true}}{\mathrm{false}}$ is obviously monic, and so it is an isomorphism. Therefore $\mathscr{F}[H]$ is Boolean, which implies, by 2.2 of [2], that H is an internal Boolean algebra.

(v)⇒(iv). Again we may without loss of generality replace $\mathscr{E}$ by $\mathscr{F}[H]$. If H is an internal Boolean algebra, then $\mathscr{F}[H]$ is Boolean by 2.2 of [2]. So the subobject classifier $1+1$ of $\mathscr{F}[H]$ is in $\mathscr{E}^*$, and is clearly a subobject classifier there as well.

Finally, if $\mathscr{E}$ is localic over $\mathscr{F}$, then $\mathscr{E}\simeq\mathscr{F}[H]$ by the relative Giraud theorem, and (v) yields (iii) by 2.2 of [2]. □

3. Final remarks

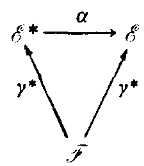
In the original version of this paper, I posed a number of open problems, one of which has recently been solved by Gordon Monro in [4]. These problems were:

(a) Does $\mathscr{E}^*\simeq\tilde{H}$ always have exponentials?

(b) If $\mathscr{E}$ is a Boolean topos, defined and localic over a Boolean topos $\mathscr{F}$, must the inclusion $\mathscr{E}^*\hookrightarrow\mathscr{E}$ be an equivalence? (By Theorem 1.3, this is true when $\mathscr{F}=\mathscr{S}$ and the axiom of choice holds. It can also be shown to hold when $\mathscr{F}$ is *any* topos satisfying the axiom of choice.)

(c) If the answer to (b) is, in general, no, find a characterization of those toposes defined over a (Boolean) topos $\mathscr{F}$ for which $\mathscr{E}=\mathscr{E}^*$.

Let me sketch a proof that a positive answer to (a) (even just for *Boolean* $\mathscr{E}$) yields the same for (b). To begin with, it is easy to show that $\mathscr{E}^*$ inherits (finite) products from $\mathscr{E}$. So if $\mathscr{E}^*$ has exponentials, where $\mathscr{E}$ is Boolean and defined over a Boolean topos $\mathscr{F}$ by a geometric morphism γ, then $\mathscr{E}^*$ is a Boolean topos with subobject classifier $2 = 1 + 1$ inherited from $\mathscr{E}$. Thus $\mathscr{E}$ and $\mathscr{E}^*$ are both localic $\mathscr{F}$-toposes and, by the relative Giraud theorem, they are both equivalent over $\mathscr{F}$ to the topos $\mathscr{F}[B]$ of internal sheaves on the internally complete Boolean algebra $B = \gamma_* 2$ in $\mathscr{F}$. Accordingly we have a commutative diagram



where α is an equivalence. If $X \in \mathscr{E}^*$ there is a diagram of the form $X \rightarrowtail \gamma^* A$ in $\mathscr{E}$ and hence in $\mathscr{E}^*$. Applying α gives $\alpha X \rightarrowtail \alpha(\gamma^* A) = \gamma^* A$. Therefore $\alpha X \in \mathscr{E}^*$. Since α is an equivalence, every object in $\mathscr{E}$ is isomorphic to one of the form αX, and hence to one in $\mathscr{E}^*$. Therefore $\mathscr{E}^* \hookrightarrow \mathscr{E}$ is an equivalence.

Now Monro has shown that the answer to (b) is, in general, no. In fact he shows that $\mathscr{E}^* \hookrightarrow \mathscr{E}$ can fail to be an equivalence even when the base topos $\mathscr{F}$ is $\mathscr{S}$, provided the axiom of choice fails in a certain (relatively consistent) way in $\mathscr{S}$. He starts with the well-known Halpern–Levy model N of set theory in which the axiom of choice fails but in which every set is totally orderable. Then he constructs a certain complete Boolean algebra B in N such that, in the corresponding Boolean extension $N^{(B)}$, with probability 1 the power set $P(R)$ of the set of real numbers is not totally orderable. Thinking of N as our base topos $\mathscr{S}$ of sets and $N^{(B)}$ as the (Boolean) topos $\mathscr{E}$ defined over $\mathscr{S}$, it follows that the object $P(R)$ of $\mathscr{E}$ cannot be (isomorphic to an object) in $\mathscr{E}^*$, for it is not hard to see that any object in $\mathscr{E}^*$ must be totally orderable. So in this case the inclusion $\mathscr{E}^* \hookrightarrow \mathscr{E}$ cannot be an equivalence. We also see that (a) fails as well: $\tilde{B}$ does not have exponentials.

Problem (c) is, however, still open.

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